Banach Space Techniques

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Schauder basis

Let E be a normed vector space.

Definition

We say that $(\xi_n)_{n=1}^{\infty}$, a nonzero sequence in E, is a **Schauder basis** for E if for each $\xi \in E$, there is a unique sequence of complex numbers $(\alpha_n)_{n=1}^{\infty}$ such that

$$\xi = \sum_{n=1}^{\infty} \alpha_n \xi_n,$$

that is, $\left(\sum_{n=1}^{k} \alpha_n \xi_n\right)_{k=1}^{\infty}$ converges in norm to ξ.

Schauder basis

If $(\xi_n)_{n=1}^{\infty}$ is a Schauder basis for a normed space E, then

$$\operatorname{span}(\xi_n)_n := \left\{ \sum_{n=1}^{k} \alpha_n \xi_n : \alpha_1, \ldots, \alpha_k \in \mathbb{C} \right\}$$

is a dense subspace of E.

Corollary

Any normed space with a Schauder basis is separable.

However, Per Enflo constructed in 1973 a separable Banach space that does not have a Schauder basis.

Examples

- **Ex.** Fix $p \in [1, \infty)$. For each $n \geq 1$, let $\delta_n : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ be given by
$$\delta_n(k) = \delta_{n,k}.$$
Then, $\delta_n \in \ell^p$ and $(\delta_n)_{n=1}^\infty$ is a Schauder basis for ℓ^p.
- **NonEx.** Let $(\delta_n)_{n=1}^\infty$ be as above and let $\delta_0 : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ be given by
$$\delta_0(k) = \frac{1}{k}$$
Then, $\delta_0 \in \ell^2$ but $(\delta_n)_{n=0}^\infty$ is **not** a Schauder basis for ℓ^2:
$$\delta_0 = \sum_{n=1}^\infty \frac{\delta_n}{n}$$

Coordinate Functionals

Fix a Schauder basis $(\xi_n)_{n=1}^{\infty}$ of a normed space E.

Definition

For each $n \geq 1$ we define a coordinate functional $\omega_n : E \to \mathbb{C}$ by

$$\omega_n\Big(\sum_{k=1}^{\infty} \alpha_k \xi_k\Big) := \alpha_n.$$

One checks that $\omega_n \in E^*$ and clearly $\omega_n(\xi_m) = \delta_{n,m}$.

Definition

For each $n \geq 1$ we define $s_n : E \to E$ by

$$s_n\Big(\sum_{k=1}^{\infty} \alpha_k \xi_k\Big) := \sum_{k=1}^{n} \alpha_k \xi_k.$$

Coordinate Functionals

Fix a Schauder basis $(\xi_n)_{n=1}^{\infty}$ of a normed space E. Then,

- $s_n \in \mathcal{L}(E)$ is an idempotent and therefore $\|s_n\| \geq 1$.
- For each $\xi \in E$ we have

$$s_n(\xi) = \sum_{k=1}^{n} \omega_k(\xi)\xi_n.$$

- For each $\xi \in E$ we have $s_n\xi \to \xi$ and $\sup_{n\geq 1} \|s_n\| < \infty$.

The constant $K_{(\xi_n)} := \sup_{n\geq 1} \|s_n\|$ is called the **basis constant** of the basis $(\xi_n)_{n=1}^{\infty}$.
Of course $K_{(\xi_n)} \geq 1$.

Basic Sequence

Let E be a normed vector space.

Definition

We say that $(\xi_n)_{n=1}^{\infty}$ is a **basic sequence** if $(\xi_n)_{n=1}^{\infty}$ is a Schauder basis for $\overline{\mathrm{span}(\xi_n)}$, its closed linear span.

Every infinite dimensional Banach space contains a basic sequence. This was shown back in 1933 by Mazur.

Basic Sequence

To recognize when a sequence of elements in a Banach space is a basic sequence we use the following test

Proposition (Grunblum's criterion)

A sequence $(\xi_n)_{n=1}^{\infty}$ of nonzero elements of a Banach space E is basic if and only if there is a positive constant K such that

$$\Big\| \sum_{k=1}^{m} \alpha_k \xi_k \Big\| \leq K \Big\| \sum_{k=1}^{n} \alpha_k \xi_k \Big\|$$

for every sequence of scalars (α_k) and all integers m, n such that $m \leq n$

Equivalent Basic Sequences.

Let E and F be Banach spaces.

Definition

Two basic sequences $(\xi_n)_{n=1}^{\infty}$ in E and $(\eta_n)_{n=1}^{\infty}$ in F are said to be **isomorphically equivalent** if for any sequence of scalars $(\alpha_n)_{n=1}^{\infty}$

$$\sum_{n=1}^{\infty} \alpha_n \xi_n \text{ converges if and only if } \sum_{n=1}^{\infty} \alpha_n \eta_n \text{ converges}$$

The closed graph theorem implies that $(\xi_n)_{n=1}^{\infty}$ and $(\eta_n)_{n=1}^{\infty}$ are isomorphically equivalent if and only if $\overline{\operatorname{span}(\xi_n)}$ and $\overline{\operatorname{span}(\eta_n)}$ are isomorphic via $\xi_n \mapsto \eta_n$.

Equivalent Basic Sequences.

Equivalently, $(\xi_n)_{n=1}^{\infty}$ and $(\eta_n)_{n=1}^{\infty}$ are equivalent if there are constants $C_1, C_2 \in (0, \infty)$ such that

$$C_1 \Big\| \sum_{n=1}^{\infty} \alpha_n \eta_n \Big\| \leq \Big\| \sum_{n=1}^{\infty} \alpha_n \xi_n \Big\| \leq C_2 \Big\| \sum_{n=1}^{\infty} \alpha_n \eta_n \Big\|$$

for all sequences of scalars $(\alpha_n)_{n=1}^{\infty}$.

Definition

When $C_1 = C_2 = 1$, we say that the basic sequences are **isometrically isomorphic**.

Equivalent Basic Sequences

There is a test to check whether a sequence is isomorphically equivalent to a given basic sequence:

Theorem (Principle of small perturbations)

Let $(\xi_n)_{n=1}^{\infty}$ be a basic sequence in a Banach space E. If $(\eta_n)_{n=1}^{\infty}$ is a sequence in E such that

$$2K_{(\xi_n)} \sum_{n=1}^{\infty} \frac{\|\xi_n - \eta_n\|}{\|\xi_n\|} = \delta < 1$$

Then $(\eta_n)_{n=1}^{\infty}$ is a basic sequence equivalent to $(\xi_n)_{n=1}^{\infty}$.

Principle of small perturbations

Sketch of Proof. Let $\omega_n : \overline{\text{span}(\xi_n)} \to \mathbb{C}$ the coordinate functionals. By Hahn Banach these maps extend to linear functionals $\omega_n : E \to \mathbb{C}$. The map $t : E \to E$ given by

$$t(\xi) = \xi + \sum_{n=1}^{\infty} \omega_n(\xi)(\eta_n - \xi_n)$$

is linear and bounded by $1 + \delta$. It's also easy to check that $\|t - 1\| < \delta < 1$, whence t is invertible. Since t restricts to an isomorphism $\overline{\text{span}(\xi_n)} \to \overline{\text{span}(\eta_n)}$, the result follows. "□"

Block Basic Sequence

Let $(\xi_n)_{n=1}^\infty$ be a basic sequence in a Banach space E.

Definition

Let $\lambda_1 < \gamma_1 < \lambda_2 < \gamma_2 < \cdots$ be an increasing sequence of integers. For each $k \geq 1$ let

$$\eta_k := \sum_{j=\lambda_k}^{\gamma_k} \beta_j \xi_j$$

be any non-zero vector in $\mathrm{span}(\xi_{\lambda_k}, \ldots, \xi_{\gamma_k})$. Then $(\eta_k)_{k=1}^\infty$ is said to be a **block basic sequence** with respect to $(\xi_n)_{n=1}^\infty$.

Block Basic Sequence

Lemma

Let $(\eta_k)_{k=1}^{\infty}$ be a block basic sequence with respect to the basic sequence $(\xi_n)_{n=1}^{\infty}$. Then, $(\eta_k)_{k=1}^{\infty}$ is a basic sequence with basic constant at most $K_{(\xi_n)}$.

Proof. We prove this using Grunblum's criterion. Let $m \leq n$,

$$\Big\|\sum_{k=1}^{m}\alpha_k\eta_k\Big\| = \Big\|\sum_{k=1}^{m}\alpha_k\sum_{j=\lambda_k}^{\gamma_k}\beta_j\xi_j\Big\| = \Big\|\sum_{j=1}^{\gamma_m}c_j\xi_j\Big\| \leq K_{(\xi_n)}\Big\|\sum_{j=1}^{\gamma_n}c_j\xi_j\Big\|$$
$$= K_{(\xi_n)}\Big\|\sum_{k=1}^{n}\alpha_k\sum_{j=\lambda_k}^{\gamma_k}\beta_j\xi_j\Big\| = K_{(\xi_n)}\Big\|\sum_{k=1}^{n}\alpha_k\eta_k\Big\|$$

where each c_j is either $\alpha_k\beta_j$ or 0. ∎

The Bessaga–Pełczyńki Selection Principle

Proposition (Bessaga–Pełczyńki Selection Principle, BPSP)

Let $(\xi_n)_{n=1}^{\infty}$ *be a Schauder basis in a Banach space* E*. Suppose* $(\upsilon_n)_{n=1}^{\infty}$ *is a sequence in* E *such that*

- $\inf_{n\in\mathbb{Z}_{>0}} \|\upsilon_n\| > 0$
- $\lim_{n\to\infty} \omega_k(\upsilon_n) = 0$ *for all* $k \in \mathbb{Z}_{>0}$

Then, $(\upsilon_n)_{n=1}^{\infty}$ *contains a subsequence that is isomorphically equivalent to some block basic sequence* $(\eta_k)_{k=1}^{\infty}$ *of* $(\xi_n)_{n=1}^{\infty}$.

Proof. "Gliding hump" argument + Principle of small perturbations. ■

The Bessaga–Pełczyńki Selection Principle

Sketch of Proof. Let $\alpha := \inf_{n\in\mathbb{Z}_{>0}} \|v_n\|$, $K := K_{(\xi_n)}$. For any $\varepsilon \in (0, \frac{1}{4})$, proceed inductively and get a subsequence $(v_{n_k})_{k=1}^{\infty}$ and a strictly increasing sequence $(\lambda_k)_{k=0}^{\infty}$ such that

$$\|s_{\lambda_{k-1}} v_{n_k}\| < \frac{\alpha\varepsilon^k}{2K} \quad \text{and} \quad \|s_{\lambda_k} v_{n_k} - v_{n_k}\| < \frac{\alpha\varepsilon^k}{2K} \ \forall\, k \geq 1$$

For each $k \geq 1$, define $\eta_k := s_{\lambda_k} v_{n_k} - s_{\lambda_{k-1}} v_{n_k}$. Then, check that

$$2K \sum_{k=1}^{\infty} \frac{\|\eta_k - v_{n_k}\|}{\|\eta_k\|} < \frac{2}{1-\varepsilon} \sum_{k=1}^{\infty} \varepsilon^k = \frac{2\varepsilon}{(1-\varepsilon)^2} < 1$$

"□"

Infinite dimensional subspaces

Proposition

Let $(\xi_n)_{n=1}^{\infty}$ be a Schauder basis in a Banach space E and F an infinite dimensional subspace of E. Then, F contains a basic sequence that's isomorphically equivalent to a block basic sequence of $(\xi_n)_{n=1}^{\infty}$.

Proof. Well, for each $n \in \mathbb{Z}_{>0}$ consider the map $\psi_n : F \to \mathbb{C}^n$ given by

$$\psi_n(v) = (\omega_1(v), \ldots, \omega_n(v))$$

Since F is infinite dimensional but $\mathbb{C}^n$ isn't, the map ψ_n has a non-trivial kernel and therefore we can choose $v_n \in F$ such that $\|v_n\| = 1$ and $\omega_j(v_n) = 0$ for $1 \leq j \leq n$.
Then, $\inf_{n \in \mathbb{Z}_{>0}} \|v_n\| > 0$ and $\lim_{n \to \infty} \omega_k(v_n) = 0$. Result now follows from the BPSP. ■

$\ell^p \not\cong \ell^q$ for $p \neq q$ in $[1, \infty)$.

Lemma

Suppose $(\eta_k)_{k=1}^{\infty}$ is a block basic sequence in ℓ^p w.r.t $(\delta_n)_{n=1}^{\infty}$ such that $\inf_k \|\eta_k\| > 0$ and $\sup_k \|\eta_k\| < \infty$. Then $(\eta_k)_{k=1}^{\infty}$ is isomorphically equivalent to $(\delta_n)_{n=1}^{\infty}$.

Proof. Let $C_1 = \inf_k \|\eta_k\|$ and $C_2 = \sup_k \|\eta_k\|$. Then, for any $m \in \mathbb{Z}_{>0}$

$$C_1^p \Big\| \sum_{k=1}^{m} \alpha_k \delta_k \Big\|_p^p \leq \sum_{k=1}^{m} |\alpha_k|^p \|\eta_k\|_p^p \leq C_2^p \Big\| \sum_{k=1}^{m} \alpha_k \delta_k \Big\|_p^p$$

Since $(\eta_k)_{k=1}^{\infty}$ is a block basic sequence w.r.t $(\delta_n)_{n=1}^{\infty}$,

$$\Big\| \sum_{k=1}^{m} \alpha_k \eta_k \Big\|_p^p = \sum_{k=1}^{m} |\alpha_k|^p \|\eta_k\|_p^p$$

■

$\ell^p \not\cong \ell^q$ for $p \neq q$ in $[1, \infty)$.

Theorem (Pitt, 1930)

Let $1 \leq p < q < \infty$ and $t \in \mathcal{L}(\ell^q, \ell^p)$. Then $\|t(\delta^n)\|_p \to 0$ as $n \to \infty$.

Proof. We easily see that $t(\delta_n) \to 0$ weakly in ℓ^p. If $\|t(\delta^n)\|_p \not\to 0$, both hypotheses of the BPSP are met. Hence, there is $(t(\delta^{n_k}))_{k=1}$ isomorphically equivalent to some block basic sequence $(\eta_k)_{k=1}^\infty$ of $(\delta_k)_{k=1}^\infty$, the basis of ℓ^p. By lemma, $(\eta_k)_{k=1}^\infty$ is isomorphically equivalent to $(\delta_k)_{k=1}^\infty$, whence $(t(\delta_{n_k}))_k$ is too. Then, there is a constant C such that

$$\|(\alpha_k)_k\|_p = \Big\| \sum_{k=1}^\infty \alpha_k \delta_k \Big\|_p \leq C \Big\| \sum_{k=1}^\infty \alpha_k t(\delta_{n_k}) \Big\|_p \leq C\|t\| \|(\alpha_k)_k\|_q$$

for all $(\alpha_k)_k \in \ell^p \subset \ell^q$. In particular, $n^{\frac{1}{p} - \frac{1}{q}} \leq C\|t\|$ for all $n \geq 1$, which is impossible because $p < q$. ∎

$\ell^p \not\cong \ell^q$ for $p \neq q$ in $[1, \infty)$.

Corollary

Let $1 \leq p < q < \infty$. Then ℓ^p is not isomorphic to ℓ^q.

Proof. Suppose that there is an isomorphism $t : \ell^q \to \ell^p$. Then, by the previous theorem

$$1 = \|\delta_n\|_q = \|t^{-1}(t(\delta_n))\|_q \leq \|t^{-1}\| \|t(\delta_n)\|_p \to 0 \text{ as } n \to \infty,$$

which is absurd. ■

Disjointly Supported sequence in $L^p[0,1]$ for $p \in [1,\infty)$

Lemma

Let $(f_n)_{n=1}^\infty$ be a sequence of norm one functions in $L^p([0,1])$. If $m(\mathrm{supp}(f_n)) \to 0$, then there is a subsequence of (f_n) that's isomorphically equivalent to a disjointly supported sequence in $L^p([0,1])$.

Sketch of Proof. Use the measure, $\mu_n(A) := \int_A |f_n|^p dm$, which is absolutely continuous with respect to m and a "gliding hump" argument to produce a disjointly supported basic sequence $(g_k)_{k=1}^\infty$ which is a equivalent to the usual basis of ℓ^p and

$$2\sum_{k=1}^\infty \frac{\|g_k - f_{n_k}\|_p}{\|g_k\|_p} < 2\sum_{k=1}^\infty \frac{4^{-k}}{\frac{3}{4}} = \frac{8}{3}\cdot\frac{1}{3} < 1.$$

The principle of small perturbations proves that $(f_{n_k})_{k=1}^\infty$ is isomorphically equivalent to $(g_k)_{k=1}^\infty$ "□"

Questions?